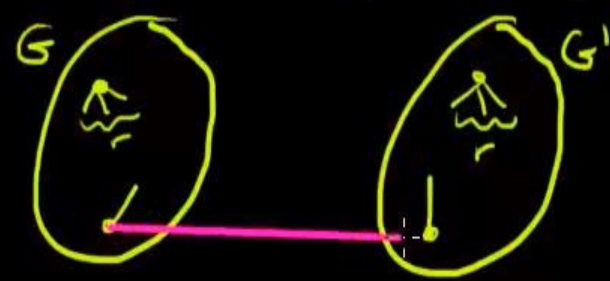
Subgraphs of regular graphs

Theorem (König): Every graph G with $\Delta(G) = r$ is an induced subgraph of an r-regular graph.

Proof: If G is r-regular, then we're done.

Suppose G is not r-regular, ie $\delta(G) < r$

Let G' be another copy of G and join corresponding vertices of G and G' with an edge if they had degree $< r$

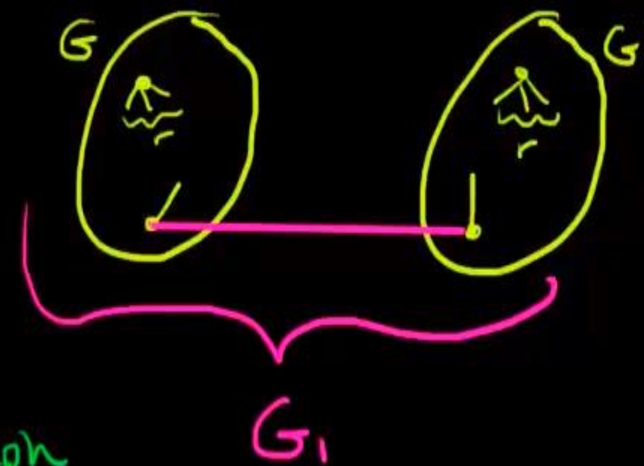


Proof: If G is r-regular, then we're done.

Suppose G is not r-regular, ie $\delta(G) < r$

Let G' be another copy of G and join corresponding vertices of G and G' with an edge if they had degree $< r$

Call the resulting graph G_1



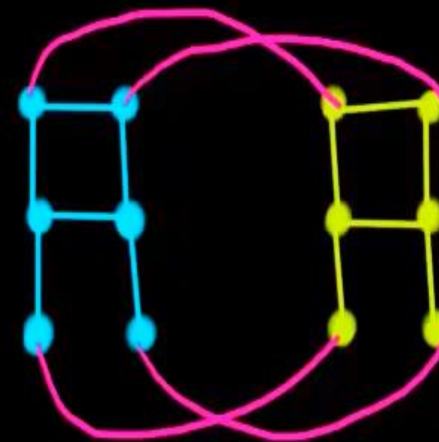
If G_1 is r-regular, then we're done since G is an induced subgraph of G_1.

If not, then continue the procedure until arriving at an r-regular graph G_K where $K = r - \delta(G)$

Ex: G

$\Delta(G) = 3$

$\delta(G) = 1$



G_1

Ex: G

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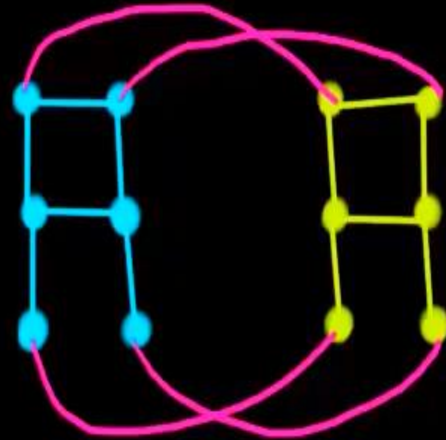
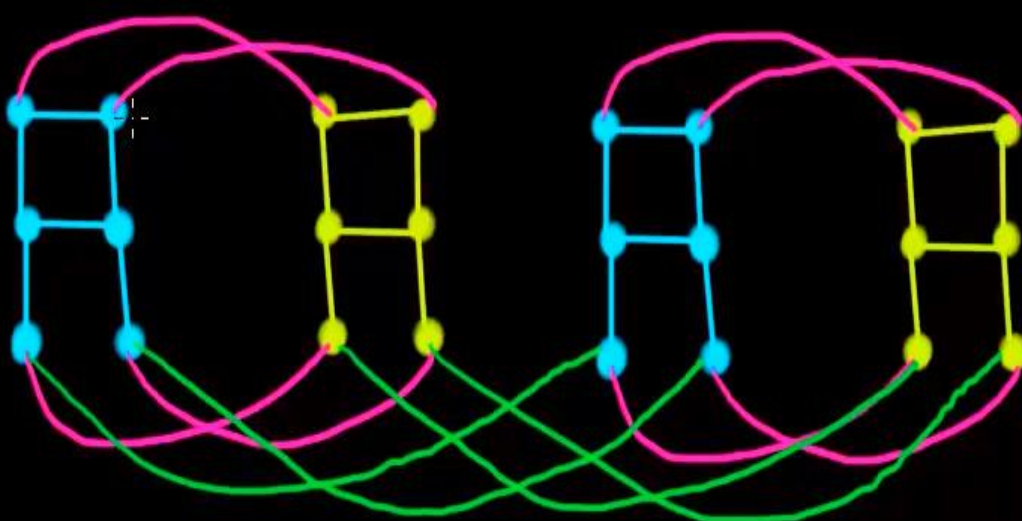


G_1

G_2 is 3-regular



G_2

Ex: G

$\Delta(G) = 3$

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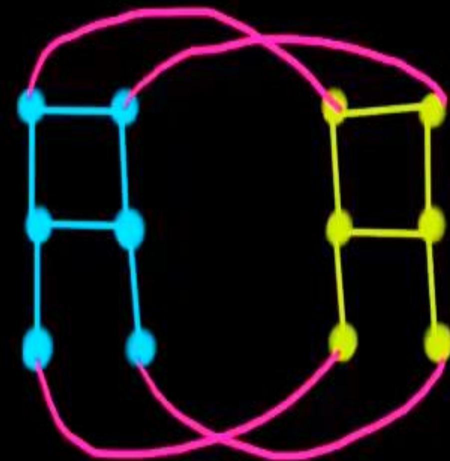

G_1

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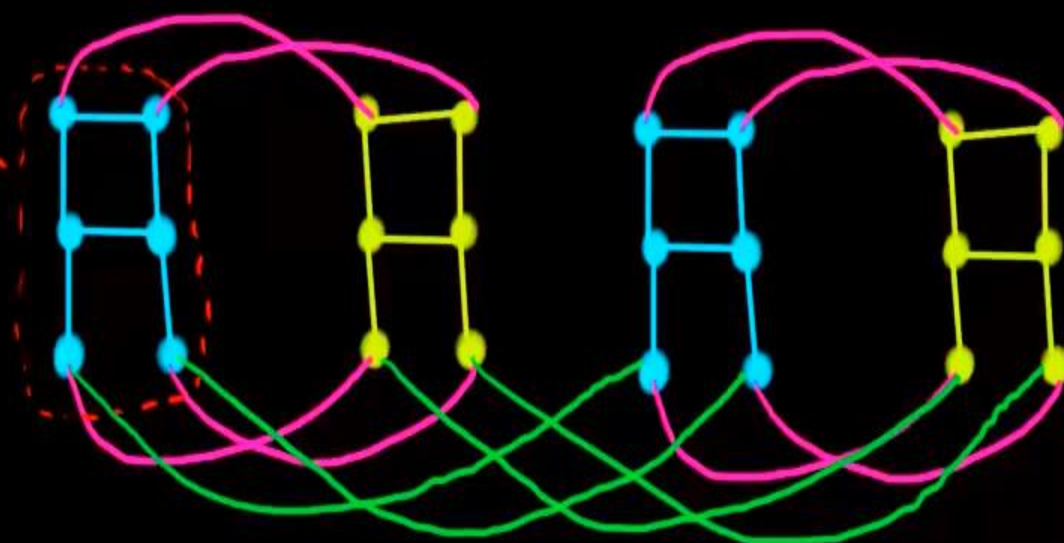
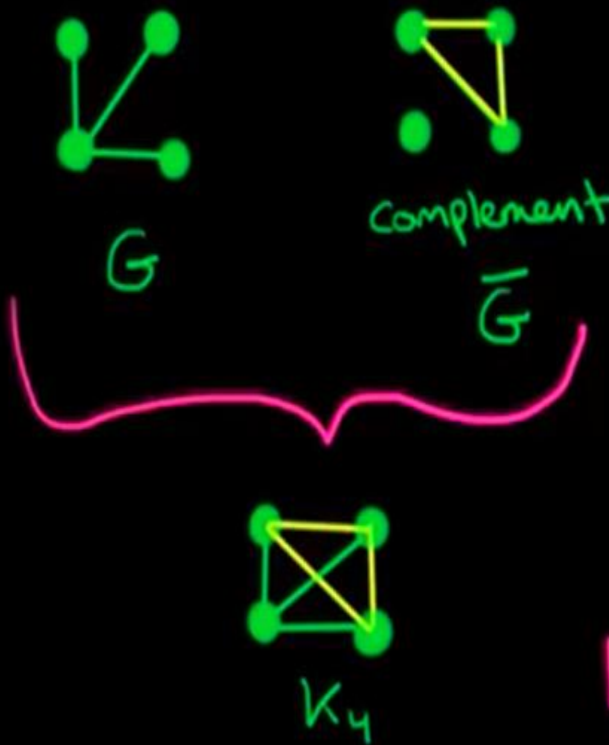


G_2

the induced subgraph on these vertices is G

Complement of a graph

Ex:



Let $G = (V, E)$ be a graph of order n. The complement graph $\overline{G}$ is the graph with

$$V(\overline{G}) = V(G)$$

and

$$E(\overline{G}) = E(K_n) \setminus E(G)$$

$$|E(\overline{G})| = \binom{n}{2} - m$$

where $m = |E(G)|$

$\binom{4}{2} = 6$

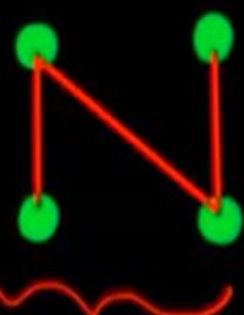
K_4

$$|E(\bar{G})| = \binom{n}{2} - m$$

where $m = |E(G)|$

A graph G is called self-complementary if $G \cong \bar{G}$

G



P_4

P_4

$\bar{G}$

G

$\bar{G}$

$G \cong \bar{G} \cong C_5$

$\binom{4}{2} = 6$ K_4

$$|E(\overline{G})| = \binom{n}{2} - m$$

where $m = |E(G)|$

A graph G is called self-complementary if $G \cong \overline{G}$

G

P_4

P_4

$\overline{G}$

$\binom{4}{2} = 6$ and $G, \overline{G}$ each have 3 edges

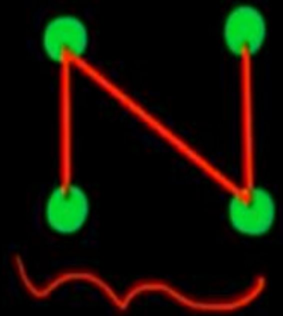
G

$\overline{G}$

$\binom{5}{2} = 10$ and $G, \overline{G}$ each have 5 edges

$G \cong \overline{G} \cong C_5$

G



P_4

$\overline{G}$

P_4

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G

$\overline{G}$

$\binom{5}{2} = 10$ and $G, \overline{G}$ each have 5 edges

$G \cong \overline{G} \cong C_5$

Question: Is there a self-complementary graph of order 6?

↳ NO since $\binom{6}{2} = 15$ is <u>odd</u>

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↳ NO since $\binom{6}{2} = 15$ is odd

Fact: A self-complementary graph of order n must have

$$\frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4} \text{ edges}$$

$\therefore$ If G of order n is self-complementary

then $n(n-1)$ must be divisible by 4

so $n \equiv 0$ or $1 \pmod 4$.

$n=4$ ✓

$n=5$ ✓

$n=6$ ✗

$\therefore$ If G of order n is self-complementary then $n(n-1)$ must be divisible by 4

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$n=4$ ✓

$n=5$ ✓

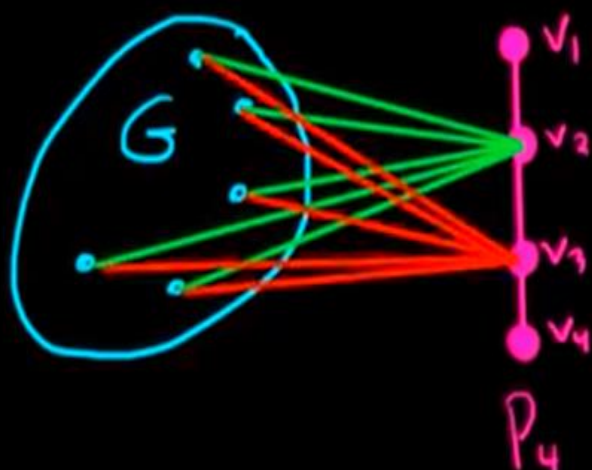
$n=6$ ✗

$n=7$ ✗

$n=8$

$n=9$

Given a self-complementary graph G of order n we can construct a self-complementary graph G' of order $n+4$ as follows:



Add $P_4 = [v_1, v_2, v_3, v_4]$ and join vertex v_2 and vertex v_3 to all vertices of G

∴ If G of order n is self-complementary then $n(n-1)$ must be divisible by 4

so $n \equiv 0$ or $1 \pmod 4$.

$n=4$ ✓ $\quad n=8$ ✓ $\quad n=12$ ✓

$n=5$ ✓ $\quad n=9$ ✓ $\quad n=13$ ✓

$n=6$ ✗

$n=7$ ✗

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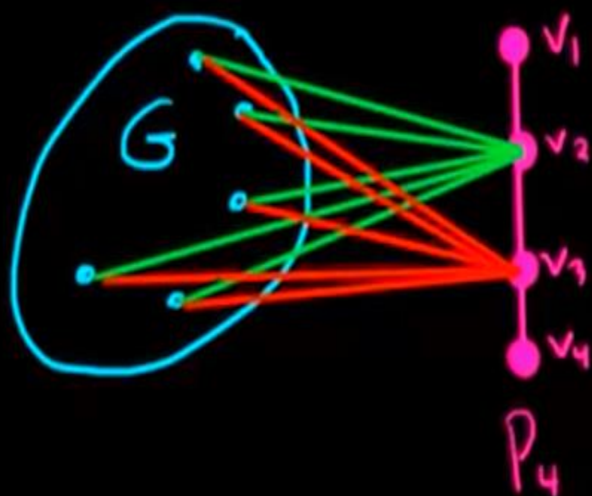

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Cartesian Product of Graphs

Recall: The cartesian product of 2 sets A and B is

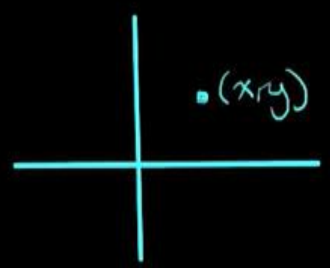
$$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$$

Ex: $A = \{x, y\}$ $B = \{1, 2, 3\}$

Then $A \times B = \{ (x,1), (x,2), (x,3), (y,1), (y,2), (y,3) \}$

Ex: Euclidean plane : $\mathbb{R} \times \mathbb{R}$

Note: $\mathbb{R}$ is the set of real numbers



$x \in \mathbb{R}$

$y \in \mathbb{R}$

Suppose G and H are graphs with $V(G) = \{u_1, u_2, \ldots, u_m\}$
and $V(H) = \{v_1, v_2, \ldots, v_n\}$

can have different # of vertices

Then $G \times H$ is the graph with vertex set

$$V(G \times H) = V(G) \times V(H)$$
$$= \{(u_i, v_j) \mid u_i \in V(G),\ v_j \in V(H)\}$$

and e is an edge of $G \times H$ <u>iff</u> $e = (u_i, v_j)(u_k, v_\ell)$

where either:

1) $i = k$ and $v_j v_\ell \in E(H)$

or 2) $j = \ell$ and $u_i u_k \in E(G)$

Ex: $K_2 \times P_4$

K_2 

P_4 

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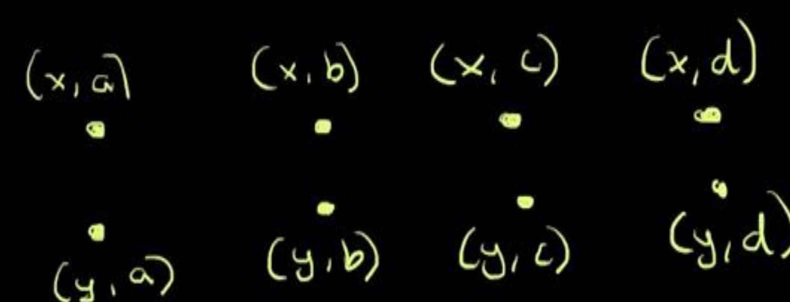

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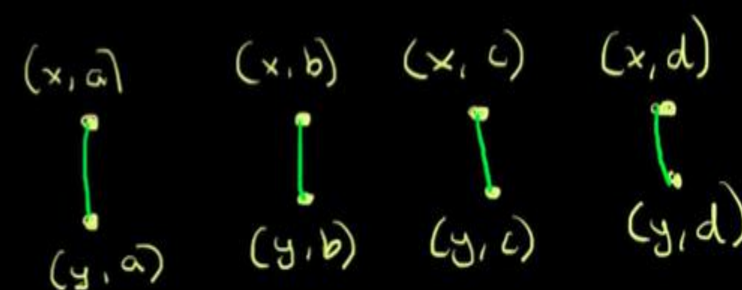

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and e is an edge of $G \times H$ **iff** $e = (u_i, v_j)(u_k, v_l)$

where either:

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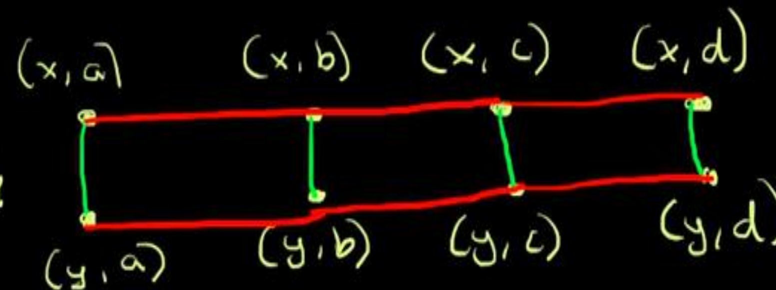
or 2) $j = l$ and $u_i u_k \in E(G)$

Ex: $K_2 \times P_4$

K_2 

P_4 

$K_2 \times P_4$ 

Ex: $K_4 \times K_3$

$= \{ (u_i, v_j) \mid u_i \in V(G), \ v_j \in V(H) \}$

and e is an edge of $G \times H$ **iff** $e = (u_i, v_j)(u_k, v_l)$

where either:

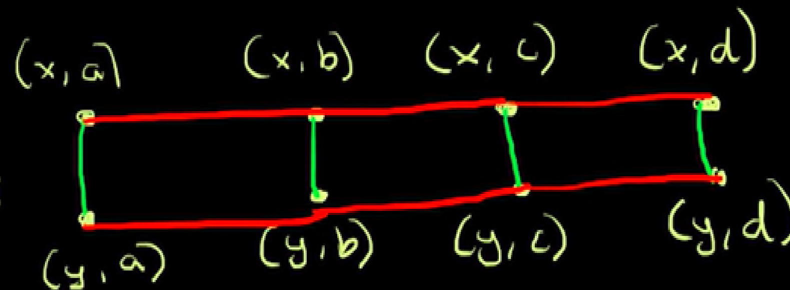
1) $i = k$ and $\underline{v_j v_l} \in E(H)$

or 2) $j = l$ and $\underline{u_i u_k} \in E(G)$

Ex: $K_2 \times P_4$

K_2 

P_4 

$K_2 \times P_4$ 

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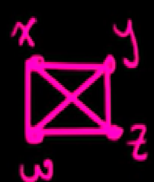
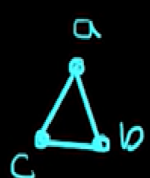
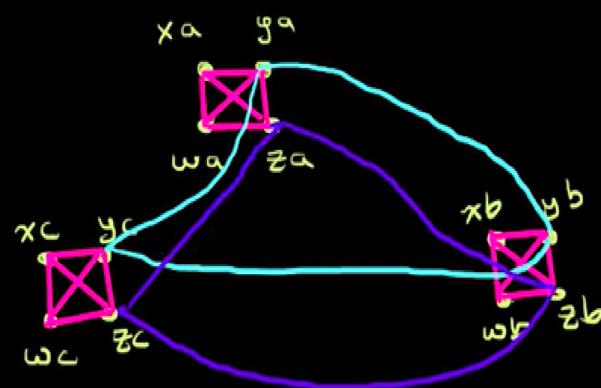






The hypercube (or n-cube) Q_n is the graph

$$Q_n = \underbrace{K_2 \times K_2 \times \cdots \times K_2}_{n \text{ times}}$$

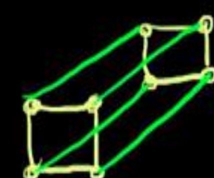
ie $Q_1 = K_2$

$Q_n = Q_{n-1} \times K_2$

Q_1

Q_2

Q_3



Gray Codes

↳ A method for listing all 0-1 sequences of length n

Naive way: using the binary (base 2) representation of the integers $0, 1, 2, \ldots, 2^n - 1$

Ex: $n = 3$

0	1	2	3	4	5	6	7
000	001	010	011	100	101	110	111

ie $3 = 2^1 + 2^0$ $\quad 6 = 2^2 + 2^1$

Another way: 000 001 101 100 110 111 011 010

differ in exactly 1 position

This is an example of a Gray code

Motivation:

$011 \to 100$

all 3 electrical switches must change states

The second listing of these sequences requires only one switch to change states

ie $101 \to 100$

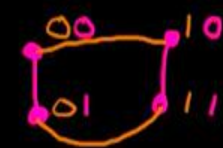
A **Gray code** is a list of the 2^n binary sequences of length n such that consecutive sequences in the list differ in exactly 1 position

A Gray code is called **cyclic** if the 1^{st} and last sequences in the list also differ in exactly 1 position.

The n-cube Q_n is the graph whose vertices are labelled by the binary sequences of length n where two vertices are adjacent if and only if their sequences differ in exactly 1 position

Ex: Q_1

Q_2



Q_3

Q_n has a recursive structure

$A \cong Q_{n-1}$

$B \cong Q_{n-1}$

"corresponding" vertices are joined

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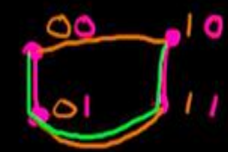
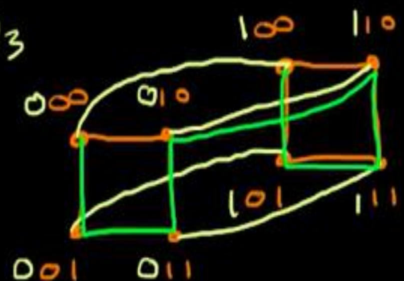


Q_3



Q_n has a recursive structure

$A \cong Q_{n-1}$

$B \cong Q_{n-1}$

"corresponding" vertices are joined

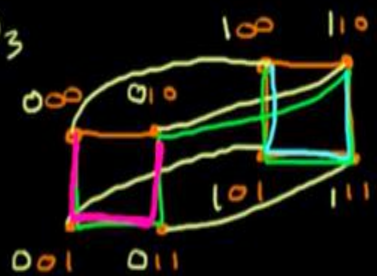
A Gray code is equivalent to a Hamilton path in Q_n

Binary Reflected Gray Codes

Q_1

Q_2

Q_3



0, 1

00, 01, 11, 10

000, 001, 011, 010, 110, 111, 101, 100

The Binary Reflected Gray Code (BRGC) is defined recursively as follows:

$$L_1 = 0, 1$$

$$L_n = 0 \cdot L_{n-1},\ 1 \cdot L_{n-1}^R \qquad \text{for } n \geq 2$$

where

$x \cdot L$ means concatenate x at the beginning of every sequence of L

L^R means L in reverse order

Fact: For each integer $n \geq 1$ the BRGC is a Gray code

↳ Proof (induction on n)

Basis: L_1 is clearly a Gray code ✓

Ind. Hyp: Suppose for some $n \geq 2$ that L_{n-1} is a Gray code

Now consider $L_n = 0 \cdot L_{n-1}, 1 \cdot L_{n-1}^R$

The sequences in $0 \cdot L_{n-1}$ still differ in 1 position (exactly) from their neighbours in the list since L_{n-1} is a Gray code

Similarly for $1 \cdot L_{n-1}^R$

Also, the first sequence in L_{n-1}^R is the last sequence in L_{n-1}

Thus the last sequence in $0 \cdot L_{n-1}$

& the first sequence in $1 \cdot L_{n-1}^R$

differ in exactly one position (the 1^{st} position)

$\therefore L_n$ is a Gray code

Thank you

Let u, v be vertices in a graph G.
The distance from u to v is the length of a shortest path from u to v in G and is denoted $d(u,v)$

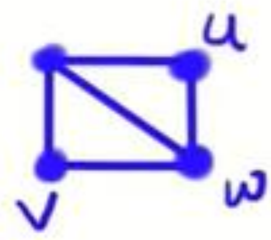
Sometimes denoted $d_G(u,v)$ for clarity

If G is disconnected and u, v are in different components we say $d(u,v) = \infty$

Ex:

$d(u,v) = 2$
$d(u,u) = 0$
$d(u,w) = 1$

Ex:



$d(u,v) = 2$
$d(u,u) = 0$
$d(u,w) = 1$

The distance is a metric space

$$d : V \times V \to \mathbb{Z}^+ \cup \{0\}$$

① $d(u,v) \geq 0$ and $d(u,v) = 0$ iff $u = v$

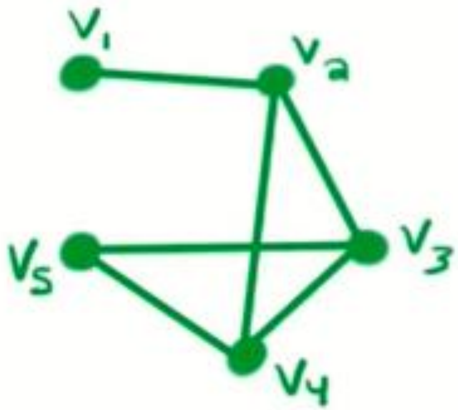
② $d(u,v) = d(v,u)$

③ $d(u,v) + d(v,w) \geq d(u,w)$

The eccentricity of a vertex $v \in V(G)$ is

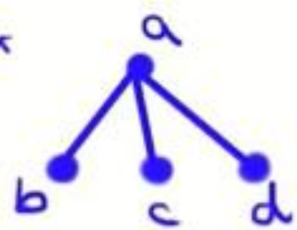
$$e(v) = \max \{ d(u,v) \mid u \in V(G) \}$$

Ex



$e(v_1) = 3$
$e(v_2) = 2$
$e(v_3) = 2$
$e(v_4) = 3$
$e(v_5) = 3$

Ex



$e(a) = 1$
$e(b) = e(c) = e(d) = 2$

Note: $e(v) = 1 \iff v$ is adjacent to all other vertices

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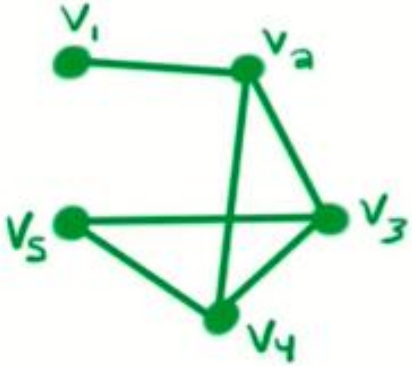
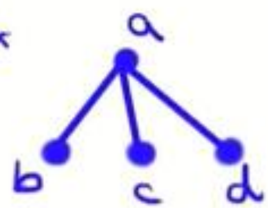


$e(v_1) = 3$
$e(v_2) = 2$
$e(v_3) = 2$
$e(v_4) = 3$
$e(v_5) = 3$

diameter 3
radius 2

Ex



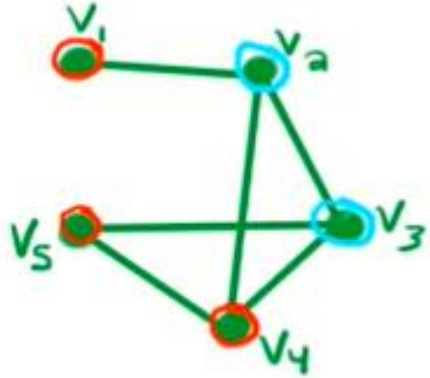
$e(a) = 1$
$e(b) = e(c) = e(d) = 2$

diameter 2
radius 1

Note: $e(v) = 1 \iff v$ is adjacent to all other vertices

$$\mathrm{diam}(G) = \max \{ e(v) \mid v \in V(G) \}$$
$$\mathrm{rad}(G) = \min \{ e(v) \mid v \in V(G) \}$$

Ex



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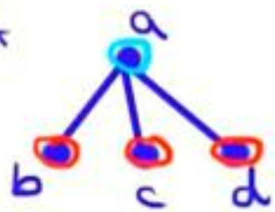
$e(v_5) = 3$

diameter 3
radius 2

Ex



$e(a) = 1$
$e(b) = e(c) = e(d) = 2$

diameter 2
radius 1

Note: $e(v) = 1 \iff v$ is adjacent to all other vertices

$\text{diam}(G) = \max\{e(v) \mid v \in V(G)\}$
$\text{rad}(G) = \min\{e(v) \mid v \in V(G)\}$

If $e(v) = \text{diam}(G)$ then v is a peripheral vertex.
The set of all such vertices make the periphery of G.

If $e(v) = \text{rad}(G)$ then v is a central vertex.
The set of all such vertices make the centre of G

Fact: $\mathrm{rad}(G) \le \mathrm{diam}(G) \le 2\,\mathrm{rad}(G)$

obvious by definition

Proof: Let $u, v \in V(G)$ such that $d(u,v) = \mathrm{diam}(G)$

G

Let w be a central vertex of G

Then

$$d(u,v) \le d(u,w) + d(w,v)$$

Fact: $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$

obvious by definition

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Let w be a central vertex of G

Then

$$
\begin{aligned}
d(u,v) &\leq d(u,w) + d(w,v) \\
&\leq 2e(w) \\
&= 2\,\mathrm{rad}(G)
\end{aligned}
$$

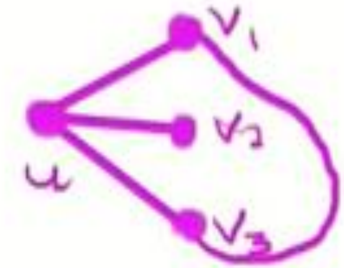
$\square$

If G is a disconnected graph, then $\overline{G}$ is connected and $\text{diam}(\overline{G}) \leq 2$

Ex:

G

$\overline{G}$



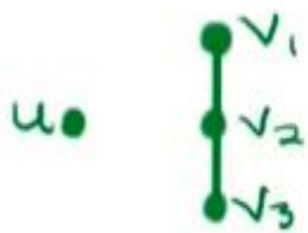
$e(u) = 1$

$e(v_1) = e(v_2)$

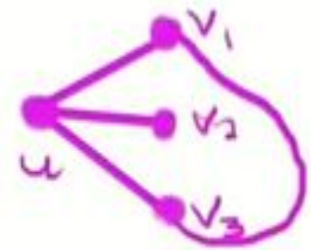
If G is a disconnected graph, then $\overline{G}$ is connected and $\text{diam}(\overline{G}) \leq 2$

Ex:

G



$\overline{G}$



$e(u) = 1$

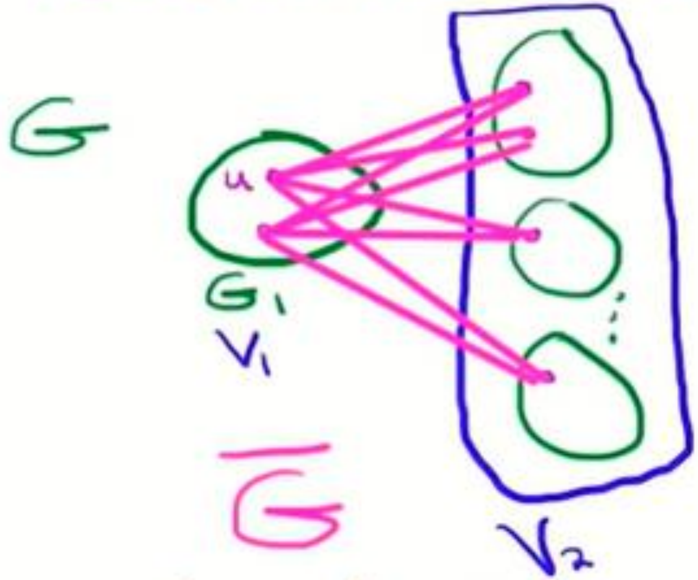
$e(v_1) = e(v_2) = e(v_3) = 2$

$\text{diam}(\overline{G}) = 2$

If G is a disconnected graph, then $\overline{G}$ is connected and $\operatorname{diam}(\overline{G}) \leq 2$

Proof:

Let G be a disconnected graph and let G_1 be one of the connected components of G



Let $V_1 = V(G_1)$

$V_2 = V(G) \setminus V(G_1)$

Let $u \in V_1$

Then for every $v \in V_2$ $uv \notin E(G)$

$\therefore uv \in E(\overline{G})$ ie $d_{\overline{G}}(u,v) = 1$

And for every $v_1, v_2 \in V_2$

$d_{\overline{G}}(v_1, v_2) \leq 2$

And for every $u_1, u_2 \in V_1$

$d_{\overline{G}}(u_1, u_2) \leq 2$

$\therefore \max\{e_{\overline{G}}(v) \mid v \in V(\overline{G})\} \leq 2$

ie $\operatorname{diam}(\overline{G}) \leq 2$

Every graph G is the centre of some connected graph.

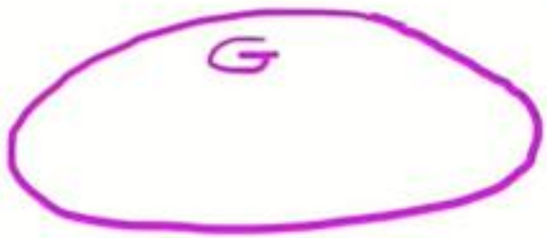
Recall: $Cen(G) = \{v \in V(G) \mid e(v) = rad(G)\}$

<u>Proof</u>:

Let G be any graph.

We construct a connected graph H as follows:

Add 2 new vertices



Let G be any graph.

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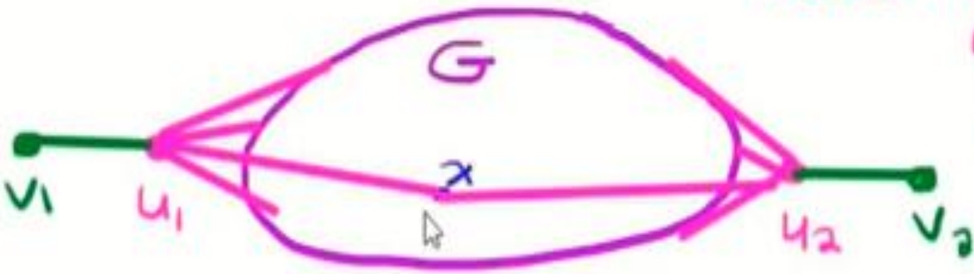


Add 2 new vertices to G
Call these u_1 & u_2
and add all edges
$u_1 x \quad \forall x \in V(G)$
and all edges
$u_2 x \quad \forall x \in V(G)$

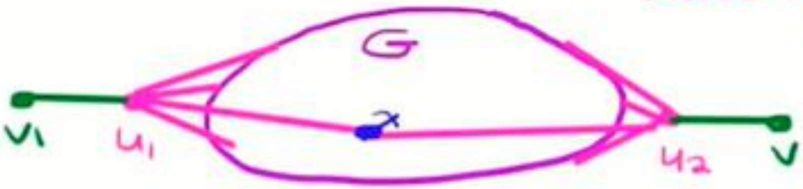
Next add 2 new vertices, v_1 and v_2 and edges $v_1 u_1$ and $v_2 u_2$

In the new graph H we have

$$e_H(v_1) = 4$$

Let G be any graph.

We construct a connected graph H as follows:



Add 2 new vertices to G
Call these u_1 & u_2
and add all edges
$u_1 x \quad \forall x \in V(G)$
and all edges
$u_2 x \quad \forall x \in V(G)$

Next add 2 new vertices, v_1 and v_2 and edges $v_1 u_1$ and $v_2 u_2$

In the new graph H we have

$$e_H(v_1) = 4 \quad \& \quad e_H(v_2) = 4$$

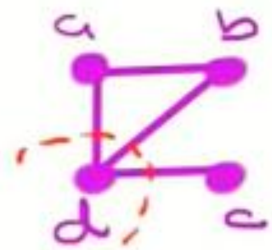
$$e_H(u_1) = 3 \quad \& \quad e_H(u_2) = 3$$

and $$e_H(x) = 2 \quad \forall x \in V(G)$$

$$\therefore \ rad(H) = 2 \quad \text{and} \quad Cen(H) = \{v \in V(H) \mid e_H(v) = rad(H)\}$$
$$= V(G)$$

connected

disconnected



connected
but easily made disconnected by removing vertex d

A vertex $v \in V(G)$ is called a cut-vertex if $G-v$ has more connected components than G

ie $c(G-v) > c(G)$

Notation: $c(G)$ = # of connected components of G
(Sometimes $\kappa(G)$ is used)

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Notation: $c(G)$ = # of connected components of G

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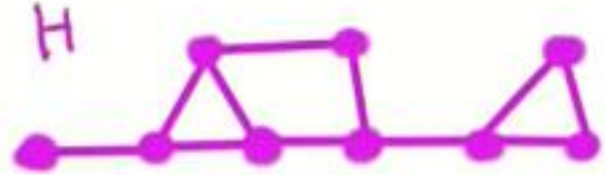
Ex: Find the cut-vertices:

G

H



A characterisation of cut-vertices:

Theorem: A vertex $v \in V(G)$ is a cut-vertex of G

$\iff \exists\, u, w \in V(G)$ $u, w \neq v$ such that v is on every u-w path of G.

Proof:

Assume G is connected (otherwise just repeat the argument for each connected component)

$\Rightarrow$] Let $v \in V(G)$ be a cut-vertex

Then $G - v$ is disconnected.

Let u, w be vertices in different components of $G - v$

So $\nexists$ any u-w path in $G - v$

But G is connected so $\exists$ u-w paths in G $\therefore$ all such paths went through vertex v

↳ Proof:

Assume G is connected (otherwise just repeat the argument for each connected component)

$\Rightarrow$] Let $v \in V(G)$ be a cut-vertex

Then $G-v$ is disconnected.

Let u, w be vertices in different components of $G-v$

So $\nexists$ any u-w path in $G-v$

But G is connected so $\exists$ u-w paths in G $\therefore$ all such paths went through vertex v

$\Leftarrow$] Suppose $\exists\, u, w \in V(G)$ $u, w \neq v$ such that v lies on every u-w path

Then removing v means $\nexists$ any u-w path in $G-v$

$\therefore$ $G-v$ is disconnected

$\therefore$ v is a cut-vertex

$\square$

How many cut-vertices can a connected graph have?

no cut-vertices

P_n

has $n-2$ cut-vertices

<u>Theorem</u>: Every non-trivial connected graph contains ≥ 2 vertices that are not cut-vertices

Theorem: Every non-trivial connected graph contains ≥ 2 vertices that are not cut-vertizes

Proof: (by Contradiction)

Suppose the theorem is NOT true

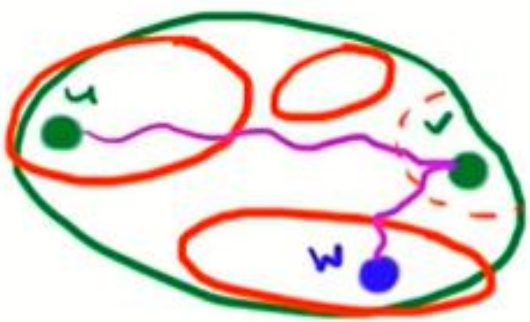
Then $\exists$ a non-trivial connected graph G with at most 1 non-cut-vertex

(ie every vertex except possibly 1 is a cut-vertex)

Let $u, v \in V(G)$ with $d(u,v) = \text{diam}(G)$

At least one of u, v is a cut-vertex, say v

So $G-v$ is disconnected



$G-v$

Let w be a vertex in a different component of $G-v$ than u

Then every $u-w$ path contains v

$\therefore d(u,w) > d(u,v) = \text{diam}(G)$

$(\Rightarrow\Leftarrow)$

*This proof actually shows that no peripheral vertex is a cut-vertex

Thank you