Wagner theorem

Graph Minor
Edge contraction
Subdivision

Recall: $K_{3,3}$ is not planar

K_5 is not planar

If G contains a non-planar subgraph, then G is non-planar

Ex: G

non-planar

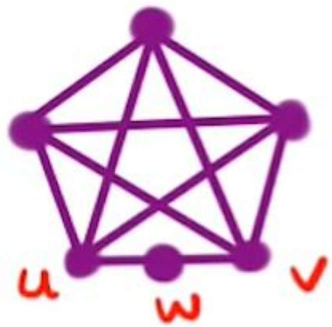
Observe

G

also non-planar

Observe

G



also non-planar

G is an elementary subdivision of K_5

An <u>elementary subdivision</u> of a nonempty graph G is a graph obtained from G by removing an edge $e = uv$ and adding a new vertex w and new edges uw and vw

G

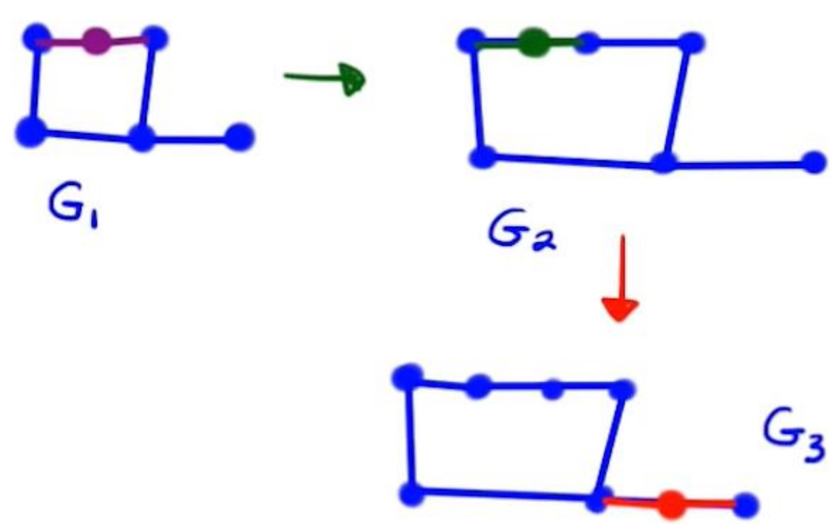
An **elementary subdivision** of a nonempty graph G is a graph obtained from G by removing an edge $e = uv$ and adding a new vertex w and new edges uw and vw

G

A **subdivision** of a graph G is a graph obtained from G by a sequence of zero or more elementary subdivisions

G

(G is a subdivision of itself)



Fact: Any subdivision H of a graph G is planar if and only if G is planar

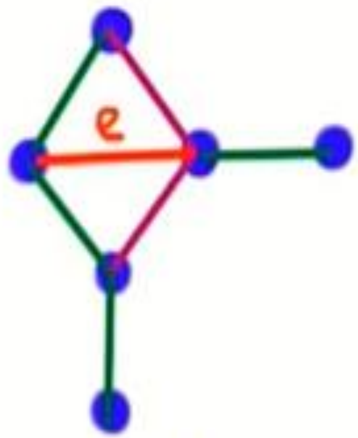
Fact: If a graph G is a subdivision of K_5 or $K_{3,3}$ then G is nonplanar

Fact: If a graph G contains a subgraph that is a subdivision of K_5 or $K_{3,3}$ then G is nonplanar

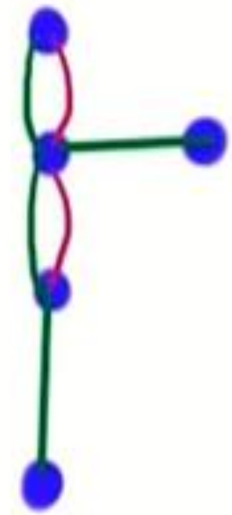
Kuratowski's Theorem:

A graph G is planar <u>if and only if</u> G contains no subgraph which is a subdivision of K_5 or $K_{3,3}$

Edge Contraction

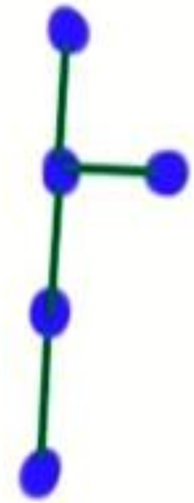


original
graph G



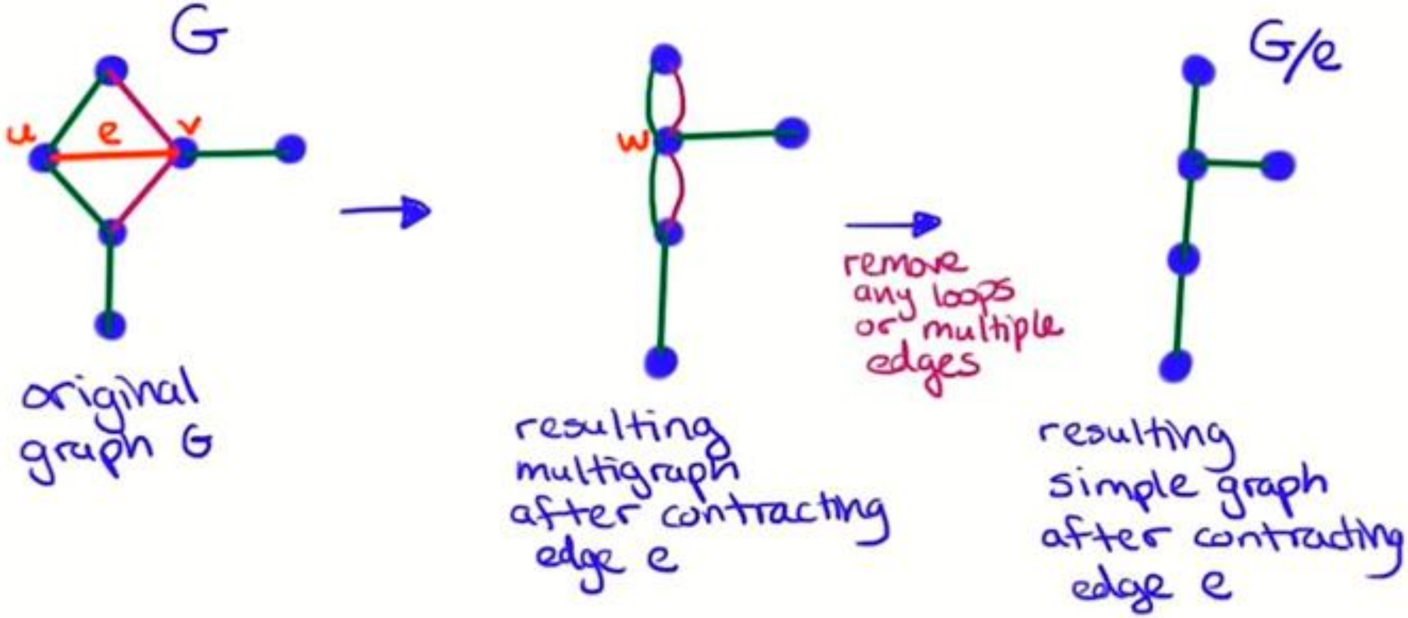
resulting
multigraph
after contracting
edge e

remove
any loops
or multiple
edges



resulting
simple graph
after contracting
edge e

Edge Contraction



Let G be a graph and $e \in E(G)$ with $e = uv$. Suppose $w \notin V(G)$. Contracting edge e in G results in the graph G/e obtained from G by:

- removing edge e
- replacing vertices u and v with a new single vertex w
- vertex w is adjacent to the neighbours of u and to the neighbours of v

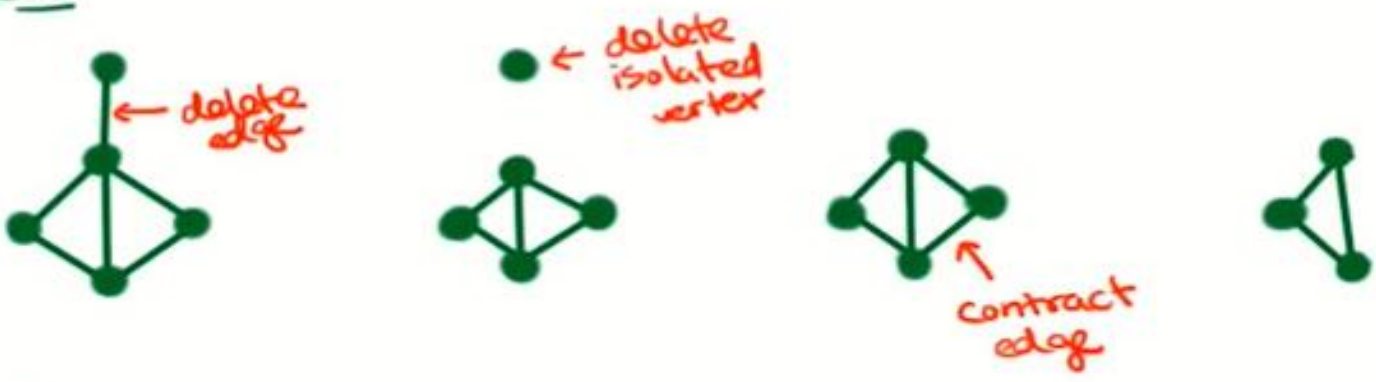
A graph H is called a <u>minor</u> of G if H can be produced from G by successive application of these reductions:

(a) deleting an edge
(b) contracting an edge
(c) deleting an isolated vertex

Note: G is a minor of itself

Every graph that is isomorphic to a minor of G is also called a minor of G.

<u>Ex</u>:



G has a K_3-minor

Wagner's Theorem: [1937]

A graph is planar $\Longleftrightarrow$ it has no K_5 or $K_{3,3}$ minor

Kuratowski's Theorem: [1930]

A graph is planar $\Longleftrightarrow$ it has no subgraph that is a subdivision of K_5 or $K_{3,3}$

Notes:

- A subdivision of H can be converted into an H-minor by contracting all but one edge in each path formed by the subdivision process

subdivision of K_3:

- <u>BUT</u> an H-minor cannot always be converted into a subdivision of H

- **BUT** an H-minor cannot always be converted into a subdivision of H

* For K_5 and $K_{3,3}$: If G has ≥ 1 of these as a minor, then H has ≥ 1 of these as a subdivision

Petersen Graph:

Fact: The Petersen graph is non-planar

Proof ① The Petersen graph contains a subgraph that is a subdivision of $K_{3,3}$

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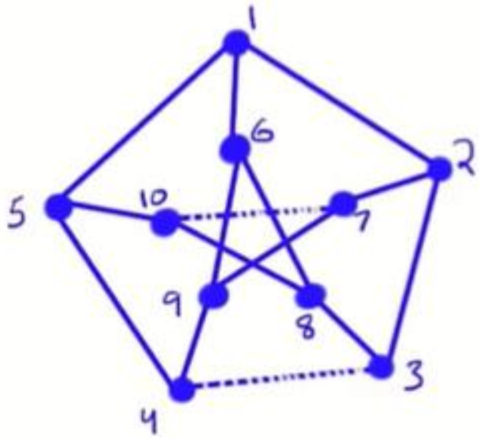

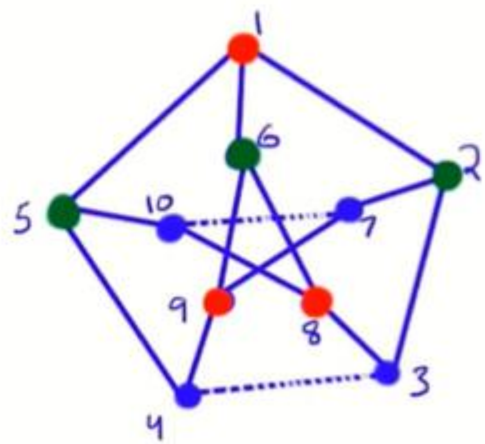

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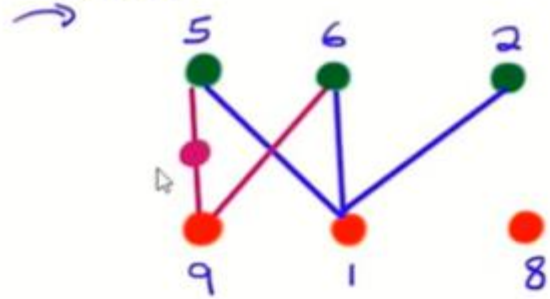
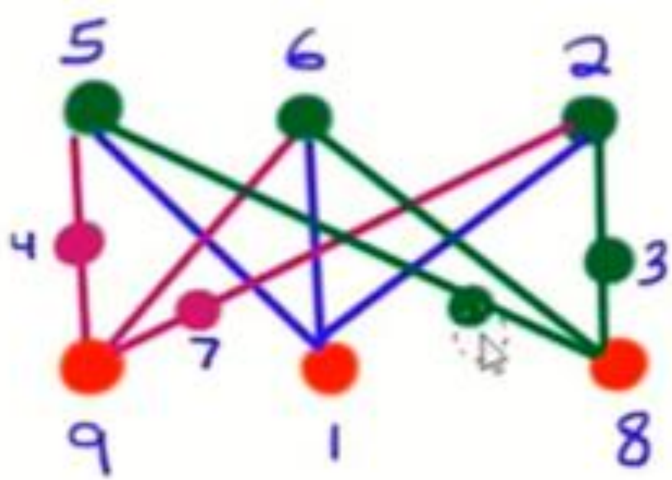
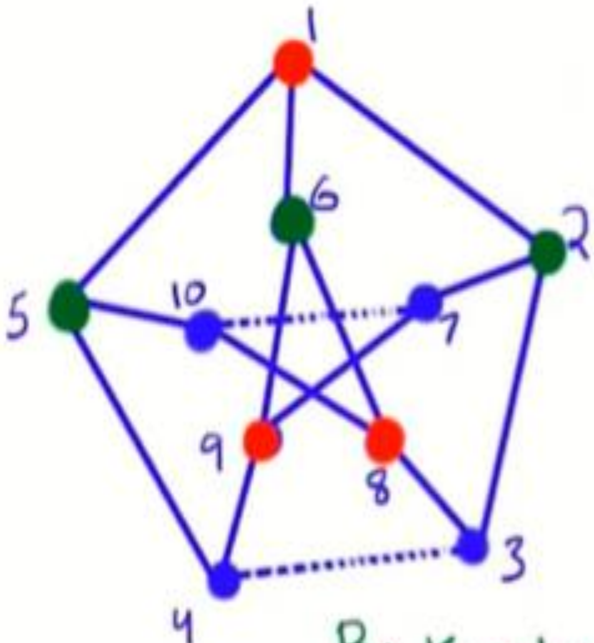




Proof ① The Petersen graph contains a subgraph that is a subdivision of $K_{3,3}$



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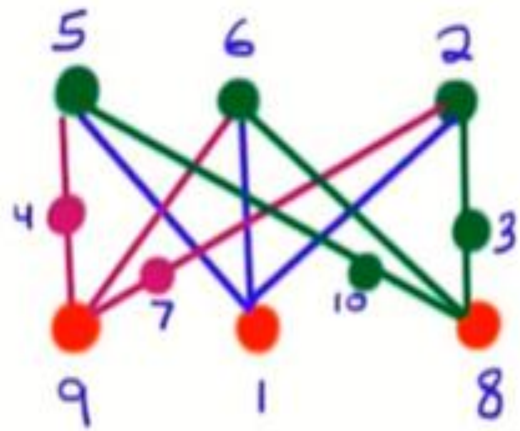
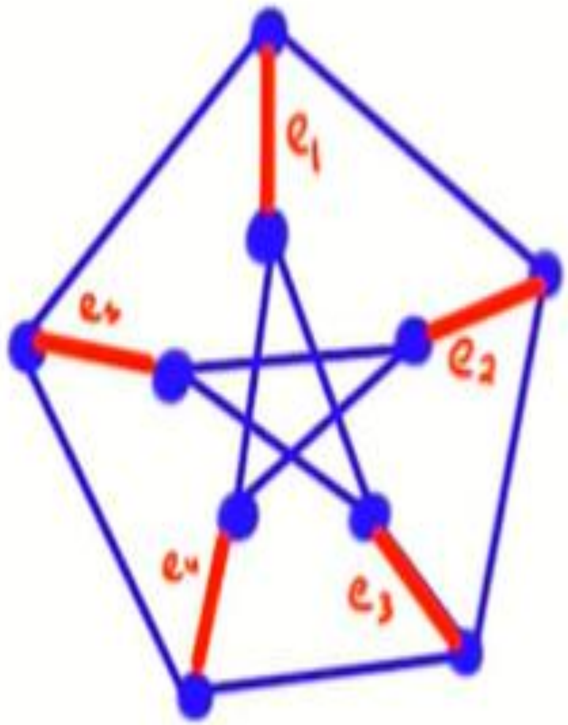


By Kuratowski's Thm, the Petersen graph is nonplanar ∎

Proof ②: The Petersen graph has a K_5 - minor



Perform edge contractions on edges e_1, e_2, e_3, e_4, e_5

The resulting graph is isomorphic to K_5

By Wagner's Thm, the Petersen graph is nonplanar

Thank you