Practice Problems: Probability and Multivariate Random Variables

1. **Sample Space and Events:** Consider rolling two fair six-sided dice. Let A be the event that the sum is 7, and B be the event that the first die shows a 4.
   (a) Describe the sample space Ω. What is $|\Omega|$?
   (b) List the outcomes in event A. Calculate $P(A)$.
   (c) List the outcomes in event B. Calculate $P(B)$.
   (d) Calculate $P(A \cap B)$ and $P(A \cup B)$.
2. **Axioms and Conditional Probability:** A box contains 5 red balls and 3 blue balls. Two balls are drawn without replacement.
   (a) What is the probability that the second ball drawn is blue, given the first ball was red?
   (b) What is the probability that both balls drawn are red?
   (c) Verify that the probability assignment satisfies the axioms of probability for the events "first ball red" and "first ball blue".
3. **Law of Total Probability:** Urn 1 contains 2 white and 3 black balls. Urn 2 contains 4 white and 1 black ball. An urn is chosen at random (equal probability), and then a ball is drawn from the chosen urn. What is the probability that the drawn ball is white?
4. **Bayes' Theorem:** Using the setup from Problem 3, suppose a white ball was drawn. What is the probability that it came from Urn 1?
5. **Independent Events:** Let A and B be two events with $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cup B) = 0.7$. Are events A and B independent? Justify your answer.
6. **Binomial Distribution:** A biased coin has a probability of landing heads (H) of $p = 0.6$. The coin is flipped 5 times independently. Let X be the number of heads obtained.
   (a) What is the probability of getting exactly 3 heads? i.e., find $P(X = 3)$.
   (b) Calculate the expected number of heads, $E[X]$.
   (c) Calculate the variance of the number of heads, $Var(X)$.
7. **Geometric Distribution:** A basketball player makes free throws with probability $p = 0.7$. Let Y be the number of attempts needed to make the first successful free throw.
   (a) Find the probability that the first success occurs on the 3rd attempt, $P(Y = 3)$.
   (b) Calculate the expected number of attempts needed, $E[Y]$.
8. **Poisson Distribution:** The number of customer arrivals at a service desk follows a Poisson distribution with an average rate of $\lambda = 4$ customers per hour.

(a) What is the probability that exactly 2 customers arrive in a given hour?

(b) What is the probability that at least one customer arrives in a given hour?

(c) What are the mean and variance of the number of arrivals per hour?

9. **Uniform Distribution:** A random variable X is uniformly distributed on the interval $[5, 15]$.

   (a) Write down the probability density function (PDF) $f_X(x)$.

   (b) Calculate the cumulative distribution function (CDF) $F_X(x)$.

   (c) Calculate $P(7 \leq X \leq 12)$.

   (d) Calculate $E[X]$ and $Var(X)$.

10. **Exponential Distribution:** The lifetime of a certain electronic component (in years) follows an exponential distribution with parameter $\lambda = 0.2$.

    (a) What is the probability density function (PDF)?

    (b) What is the probability that the component lasts more than 5 years? $P(X > 5)$.

    (c) What is the expected lifetime $E[X]$?

    (d) Using the memoryless property, what is the probability that a component that has already lasted 5 years will last at least another 3 years? $P(X > 8|X > 5)$.

11. **Normal Distribution:** Let $X \sim N(\mu = 10, \sigma^2 = 4)$.

    (a) Standardize the random variable X to $Z \sim N(0, 1)$.

    (b) Express $P(8 \leq X \leq 11)$ in terms of the standard normal CDF $\Phi(z)$. (You don't need to find the numerical value).

    (c) What is $E[2X + 5]$ and $Var(2X + 5)$?

12. **Expected Value and Variance:** A discrete random variable X has the following PMF: $P(X = 0) = 0.2$, $P(X = 1) = 0.5$, $P(X = 2) = 0.3$.

    (a) Calculate $E[X]$.

    (b) Calculate $E[X^2]$.

    (c) Calculate $Var(X)$ using the formula $Var(X) = E[X^2] - (E[X])^2$.

    (d) Calculate $E[5X - 3]$ and $Var(5X - 3)$.

13. **Bivariate Discrete Distribution:** Two discrete random variables X and Y have the following joint PMF $p_{X,Y}(x, y)$:

	$y = 0$	$y = 1$	$p_X(x)$
$x = 0$	0.1	0.2	?
$x = 1$	0.3	0.4	?
$p_Y(y)$	?	?	1

    (a) Find the marginal PMFs $p_X(x)$ and $p_Y(y)$.

    (b) Find the conditional PMF $p_{Y|X}(y|X = 1)$.

    (c) Calculate $E[XY]$.

    (d) Calculate $Cov(X, Y)$.

    (e) Are X and Y independent?

14. **Bivariate Continuous Distribution:** Let X and Y be continuous random variables with joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} c(x+y) & \text{if } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

  (a) Find the value of the constant c that makes this a valid PDF.
  (b) Find the marginal PDF $f_X(x)$.
  (c) Find the conditional PDF $f_{Y|X}(y|x)$.
  (d) Are X and Y independent?

15. **Independence Check:** Suppose the joint PDF of X and Y is given by $f_{X,Y}(x,y) = e^{-(x+y)}$ for $x > 0, y > 0$, and 0 otherwise. Find the marginal PDFs $f_X(x)$ and $f_Y(y)$ and determine if X and Y are independent.

16. **Covariance and Correlation:** Given $E[X] = 2$, $E[Y] = 3$, $Var(X) = 4$, $Var(Y) = 9$, and $E[XY] = 7$. Calculate the covariance $Cov(X,Y)$ and the correlation coefficient $\rho_{X,Y}$. Are X and Y uncorrelated?

17. **Conditional Expectation:** Using the joint PMF from Problem 13, calculate the conditional expectation $E[Y|X=0]$.

18. **Law of Total Expectation:** Suppose $E[Y|X=x] = 2x+1$ and X is a random variable with $E[X] = 5$. Calculate $E[Y]$ using the Law of Total Expectation.

19. **Multivariate Normal Distribution:** Let $\mathbf{X} = [X_1, X_2]^T$ follow a bivariate normal distribution with mean vector $\boldsymbol{\mu} = [1,2]^T$ and covariance matrix

$$\mathbf{\Sigma} = \begin{pmatrix} 4 & 1 \\ 1 & 9 \end{pmatrix}$$

  (a) What are $E[X_1]$, $E[X_2]$, $Var(X_1)$, $Var(X_2)$, and $Cov(X_1, X_2)$?
  (b) Are X_1 and X_2 independent? Why or why not?
  (c) What is the distribution of the linear combination $W = X_1 + 2X_2$? (Specify mean and variance).

20. **Multinomial Distribution:** An experiment consists of rolling a fair six-sided die 10 times. Let X_1 be the number of times '1' appears, X_2 be the number of times '2' or '3' appears, and X_3 be the number of times '4', '5', or '6' appears.

  (a) What are the parameters n, k, and the probabilities p_1, p_2, p_3 for the multinomial distribution of (X_1, X_2, X_3)?
  (b) Calculate the probability of observing the outcome $(x_1 = 2, x_2 = 3, x_3 = 5)$.